
Operational Semantics of the Lambda Calculus

A small language

Expressions

$$a ::= n \mid X \mid a + a$$

Environments are functions from variables to integers, they are denoted by σ .

We want to evaluate an expression a w.r.t. an environment σ .

Defining an Operational Semantics

- Granularity
- Order of evaluation

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Big-step Semantics

Each rule **completely** evaluates an expression w.r.t. an environment to a **value**.

$$\frac{}{n, \quad n} \quad \frac{}{X, \quad (X)}$$

$$\frac{a_1, \quad n_1 \quad a_2, \quad n_2}{a_1 + a_2, \quad n}$$

where n is the sum of n_1 and n_2

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Properties

- Abstract
- Allows to avoid details.
- No specification of evaluation order (e.g. $(1 + 3) + (5 - 3)$).
- No specification of control of errors.
- No specification of interleaving.

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Properties

- Less abstract
- Specification of order of evaluation
- Control of errors : $\frac{n_2 = 0}{n_1/n_2 \rightsquigarrow n}$, where n is n_1 divided by n_2 .
- Interleaving : $\frac{c_1, \rightsquigarrow c_1,}{c_1//c_2, \rightsquigarrow c_1//c_2,}$

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Small-step Semantics

Describes evaluation as a sequence of *state changes* of an abstract machine. Evaluation terminates when the state cannot be reduced further.

$$\frac{}{X, \rightsquigarrow (X),} \quad \frac{}{n_1 + n_2, \rightsquigarrow n,} \text{ where } n = n_1 + n_2$$

$$\frac{a_1, \rightsquigarrow a_1,}{a_1 + a_2, \rightsquigarrow a_1 + a_2,} \quad \frac{a_2, \rightsquigarrow a_2,}{n_1 + a_2, \rightsquigarrow n_1 + a_2,}$$

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From Small-step to Multi-step Semantics

Notation : $t \rightsquigarrow^* t$

- $t \rightsquigarrow^* t$ for every t
- $t \rightsquigarrow^* t$ implies $t \rightsquigarrow^* t$
- $t \rightsquigarrow^* t$ and $t \rightsquigarrow^* t$ implies $t \rightsquigarrow^* t$

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Normal Forms

- A *normal form* is a term that cannot be evaluated any further.
- A *normal form* is a state where the abstract machine is halted (result of the evaluation).
- The *meaning* of a term t in a small-step semantics is a term t such that $t \rightsquigarrow t$ and t is a normal form.

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Big-step versus Small-step Semantics

- In small-step semantics evaluation stops at errors. In big-step semantics errors occur deeply inside derivation trees.
- The order of evaluation is *explicit* in small-step semantics but *implicit* in big-step semantics.
- Big-step semantics is more abstract, but less precise.
- Small-step semantics allows to make difference between non-termination and "getting stuck".

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Properties of the Small-step Semantics

- $t \rightsquigarrow v$ iff $t \rightsquigarrow v$
- The relation \rightsquigarrow is deterministic.

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A functional language

Expressions are closed terms.

Values are **closed** terms of the form $\lambda x.M$.

We want to evaluate an expression M into a value V .

$M \rightarrow_v V$: big-step semantics for call-by-value

$M \rightarrow_n V$: big-step semantics for call-by-name

$M \rightsquigarrow_v N$: small-step semantics for call-by-value

$M \rightsquigarrow_n N$: small-step semantics for call-by-name

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Call-by-value lambda-calculus (big-step semantics)

$$\frac{}{x.M \Downarrow_v x.M}$$

$$\frac{M \Downarrow_v x.L \quad N \Downarrow_v V_2 \quad L\{x/V_2\} \Downarrow_v V_1}{MN \Downarrow_v V_1}$$

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Call-by-value lambda calculus (small-step semantics)

$$\frac{}{(x.M) V \rightsquigarrow_v M\{x/V\}}$$

$$\frac{M \rightsquigarrow_v M}{MN \rightsquigarrow_v M N} \quad \frac{N \rightsquigarrow_v N}{V N \rightsquigarrow_v V N}$$

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Examples

Let $\Delta = x.xx$. Then,

- $(x.xx)(I I) \rightsquigarrow_v (x.xx) I \rightsquigarrow_v I I \rightsquigarrow_v I$.
- $(x.I)(\Delta \Delta) \rightsquigarrow_v (x.I)(\Delta \Delta) \rightsquigarrow_v \dots$
- $\Delta \Delta \rightsquigarrow_v \Delta \Delta \rightsquigarrow_v \Delta \Delta \dots$

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Call-by-name lambda-calculus (big-step semantics)

$$\frac{}{x.M \Downarrow_n x.M}$$

$$\frac{M \Downarrow_n x.L \quad L\{x/N\} \Downarrow_n V}{MN \Downarrow_n V}$$

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Call-by-name lambda calculus (small-step semantics)

$$\frac{}{(\lambda x.M) N \rightsquigarrow_n M\{x/N\}}$$

$$\frac{M \rightsquigarrow_n M}{M N \rightsquigarrow_n M N}$$

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Coherence of results

- If $M \rightsquigarrow_v N$, then N is a value.
- If $M \rightsquigarrow_n N$, then N is a value.

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Examples

- $(\lambda x.x x) (I I) \rightsquigarrow_n (I I) (I I) \rightsquigarrow_n I (I I) \rightsquigarrow_n I I \rightsquigarrow_n I$.
- $(\lambda x.I) (\Delta \Delta) \rightsquigarrow_n I$.
- $\Delta \Delta \rightsquigarrow_n \Delta \Delta \rightsquigarrow_n \Delta \Delta \dots$

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Deterministic properties

- If $M \rightsquigarrow_v V$ and $M \rightsquigarrow_v V$, then $V = V$.
- If $M \rightsquigarrow_n P$ and $M \rightsquigarrow_n P$, then $P = P$.
- If $M \rightsquigarrow_v N$ and $M \rightsquigarrow_v N$, then $N = N$.
- If $M \rightsquigarrow_n N$ and $M \rightsquigarrow_n N$, then $N = N$.

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Relating big and small-steps semantics (i)

Lemma : If $M \rightarrow_v V$, then $M \rightsquigarrow_v V$.

Proof. By induction on $M \rightarrow_v V$.

- If $M = x.K \rightarrow_v x.K = V$, then $M \rightsquigarrow_v V$ trivially holds.
- If $M = M_1 M_2 \rightarrow_v V$ comes from $M_1 \rightarrow_v x.K$, $M_2 \rightarrow_v W$ and $K\{x/W\} \rightarrow_v V$, then $M_1 \rightsquigarrow_v x.K$, $M_2 \rightsquigarrow_v W$ and $K\{x/W\} \rightsquigarrow_v V$ hold by the inductive hypothesis so that we construct the following small-steps reduction sequence :

$$M = M_1 M_2 \rightsquigarrow_v (x.K) M_2 \rightsquigarrow_v (x.K) W \rightsquigarrow_v K\{x/W\} \rightsquigarrow_v V$$

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Taking

Relating big and small-steps semantics (ii)

Lemma : If $M \rightsquigarrow_v N$ in n steps, M is not a value and N is a value, then M is an application TU and $n_1, n_2, n_3 < n$ such that

$$M \rightsquigarrow_v^{n_1} (x.K)U \rightsquigarrow_v^{n_2} (x.K)W \rightsquigarrow_v K\{x/W\} \rightsquigarrow_v^{n_3} N$$

Proof. Suppose $M \rightsquigarrow_v N$ in n steps. We reason by induction on n .

- If $n = 0$, then $M = N$ and thus M is a value. The property holds because the hypothesis is false.
- If $n > 0$, then there are three cases.

1. $M = TU \rightsquigarrow_v T U \rightsquigarrow_v^{n-1} N$, where $T \rightsquigarrow_v T$.

Since TU is not a value we can apply the i.h. Thus

$$TU \rightsquigarrow_v^{k_1} (x.K)U \rightsquigarrow_v^{k_2} (x.K)W \rightsquigarrow_v K\{x/W\} \rightsquigarrow_v^{k_3} N.$$

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Relating big and small-steps semantics (iii)

Lemma : If $M \rightsquigarrow_v N$ and N is a value, then $M \rightarrow_v N$.

Proof. Suppose $M \rightsquigarrow_v N$ in n steps. We reason by induction on n .

- If $n = 0$, then $M = N$. But $N = x.K$ since N is a value so that $M = x.K \rightarrow_v x.K = N$.
- If $n > 0$, then M is not a value, so that by previous Lemma

$$M = TU \rightsquigarrow_v^{n_1} (x.K)U \rightsquigarrow_v^{n_2} (x.K)W \rightsquigarrow_v K\{x/W\} \rightsquigarrow_v^{n_3} N$$

for $n_1, n_2, n_3 < n$. By the i.h. $T \rightarrow_v x.K$ and $U \rightarrow_v W$ and $K\{x/W\} \rightarrow_v N$, so that we conclude $M \rightarrow_v N$.

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Relating big and small-steps semantics (iv)

- If $M \rightarrow_n P$, then $M \rightsquigarrow_n P$.
- If $M \rightsquigarrow_n N$ and N is a value, then $M \rightarrow_n N$.

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Progress properties

Let M be a closed term which is not still a value. Then,

- There exist N such that $M \rightsquigarrow_v N$.
- There exist N such that $M \rightsquigarrow_n N$.

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