

## Abstract Rewriting

### Formal definition of rewriting

Given a set of **objects**  $A$ , an (**abstract**) rewriting system is a **relation**  $R \subseteq A \times A$ .

**Example :**

$A$  = the set of finite sequences over  $\{ \cdot, \bullet, \circ \}$ .

$$\text{Rewriting system } R = \left\{ \begin{array}{ccc} \bullet & 1 & \bullet \\ \circ \bullet & 2 & \bullet \circ \\ \circ & 3 & \circ \end{array} \right.$$

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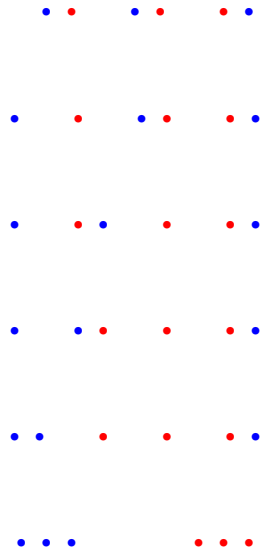
### Closure notions

- An  $R$ -rewrite sequence has the form  $s = s_0 \xrightarrow{R} s_1 \xrightarrow{R} \dots \xrightarrow{R} s_n = t$  (for  $n \geq 0$ ).
- $s \xrightarrow{R^+} t$  is the transitive closure of  $R$ .
- $\overline{R}$  is the reflexive closure.
- $\overline{R^+}$  is the reflexive transitive closure of  $R$ .
- $R^s$  is the symmetrique closure.
- $R^{st}$  is the reflexive, symetrique and transitive closure.

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### Example of $R$ -rewrite sequence

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## Different meaning for equivalent terms

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Given again

$$R = \begin{cases} f(x, x) & c \\ a & b \\ f(x, b) & d \end{cases}$$

we can compute from the same term  $f(a, a)$  two different results  $c$  and  $d$ .

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## More basic vocabulary

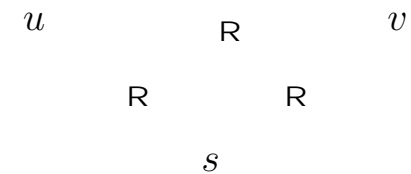
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– A term

## Same meaning for equivalent terms

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$R$  is Church-Rosser iff



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## Confluence diagrams

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A diagram like :

$$\begin{array}{ccccc} & t & & R_1 & u \\ & R_2 & & & R_3 \\ & v & & R_4 & s \end{array}$$

reads :

for all  $t, u, v$  such that  $t R_1 u$  and  $t R_2 v$ ,  
there exist  $s$  such that  $u R_3 s$  and  $v R_4 s$ .

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–  $R$  is **strongly confluent** iff

$$\begin{array}{ccccc} & t & & R & u \\ & R & & & R \\ & v & & \bar{R} & s \end{array}$$

–  $R$  has the **diamond property** iff

$$\begin{array}{ccccc} & t & & R & u \\ & R & & & R \\ & v & & R & s \end{array}$$

This is a particular case of strongly confluence.

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## Confluence notions

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–  $R$  is **confluent** iff

$$\begin{array}{ccccc} & t & & R & u \\ & R & & & R \\ & v & & R & s \end{array}$$

–  $R$  is **locally confluent** iff

$$\begin{array}{ccccc} & t & & R & u \\ & R & & & R \\ & v & & R & s \end{array}$$

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## Equivalent notions

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**Theorem** :  $R$  is **Church-Rosser** iff  $R$  is **confluent**.

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## Not equivalent notions

The following system Curry :

$$R = \left\{ \begin{array}{ll} c & a \\ c & d \\ d & c \\ d & b \end{array} \right.$$

is **locally confluent** but not **confluent** :

$$a \quad c \quad b$$

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## Weak vs strong normalisation

$$R = \left\{ \begin{array}{ll} f(a) & c \\ f(x) & f(a) \end{array} \right.$$

The system is weakly normalising but not strongly normalising :

$$f(b) \quad f(a) \quad c$$

$$f(b) \quad f(a) \quad f(a) \dots$$

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## Termination notions

- The **element**  $s$  is **R-weakly normalising (WN)** iff  $s$  has at least one normal form.
- The **element**  $s$  is **R- strongly normalising (SN)** iff there is no infinite sequence  $S = S_0 \rightarrow_R S_1 \rightarrow_R \dots$  iff every R-reduction sequence starting at  $s$  is finite. We note  $s \rightarrow_{SN} S$ .
- The **system**  $R$  is **weakly normalising (WN)** iff every element is WN.
- The **system**  $R$  **terminates** or is **strongly normalising (SN)** or **noetherien** or **well-founded (WF)** iff every element  $s$  is SN.

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## Convergent Systems

**Définition :** The **system**  $R$  is **convergent** iff it is confluent and strongly normalising.

**Remarque :**

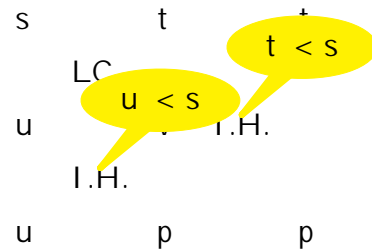
- If  $R$  is confluent, then every element has **at most** a normal form.
- If  $R$  is convergent, then every element has **one and only one** normal form. In this case, we use the *functional* notation  $R(t)$  to denote the R-normal form of  $t$ .

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## Confluence from local confluence

**Lemma : (Newmann)** Let  $R$  be a **SN** system. Then  $R$  is locally confluent iff  $R$  is confluent.

*Proof.* (By Huet) By well-founded induction on  $s \rightarrow^* \text{SN}$ .



## Important remark

The following (infinite) system on natural numbers :

$$R = \begin{cases} 2.n & 2.n + 1 \\ 2.n & \mathbf{a} \\ 2.m + 1 & 2.m + 2 \\ 2.m + 1 & \mathbf{b} \end{cases}$$

is **locally confluent** but not **confluent** :  $\mathbf{a} \rightarrow 0 \rightarrow \mathbf{b}$

In fact it is not SN

0 1 2 3 ...