
Strategies

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Classification

- Strategies without history
 - Innermost
 - Leftmost-innermost
 - Parallel-innermost
 - Leftmost-outermost (standard)
 - Parallel-outermost
 - Complete
- Strategies with history
 - Complete development
 - Standard

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Strategy

Définition : A **reduction strategy** (one-steps or multi-step) for a rewriting system R is a function $\mathcal{IF} : T(X, \Sigma) \rightarrow T(X, \Sigma)$ s.t.

1. $\mathcal{IF}(t) = t$ if t is in R -normal form.
2. $t \rightarrow^+ \mathcal{IF}(t)$ otherwise.

\mathcal{IF} is **normalizing** iff for every WN term t there is no infinite sequence $t \rightarrow \mathcal{IF}(t) \rightarrow \mathcal{IF}(\mathcal{IF}(t)) \rightarrow \mathcal{IF}(\mathcal{IF}(\mathcal{IF}(t))) \rightarrow \dots$

In what follows, we will focus only on **orthogonal** systems.

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Strategies without history

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Innermost Strategy(s)

The **innermost** strategy rewrites ONE redex of the set of all the innermost redexes.

Example :

$$R \quad \begin{array}{cc} f(a, x) & x & f(b, x) & b \\ g(a, x) & a & g(b, x) & x \end{array}$$

$$f(f(a, f(a, b)), g(\textcolor{red}{f(a, b)}, g(b, a)))$$

$$f(f(a, f(a, b)), g(f(a, b), \textcolor{red}{g(b, a)}))$$

$$f(f(a, \textcolor{red}{f(a, b)}), g(f(a, b), g(b, a)))$$

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Parallel-innermost strategy

The **parallel-innermost** strategy rewrites simultaneously ALL the innermost redexes.

$$f(f(a, \textcolor{red}{f(a, b)}), g(\textcolor{red}{f(a, b)}, \textcolor{red}{g(b, a)}))$$

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Leftmost-innermost strategy

The **leftmost-innermost** strategy rewrites the leftmost redex of the set of the innermost redexes.

Example :

$$R \quad \begin{array}{cc} f(a, x) & x \\ f(b, x) & b \\ g(a, x) & a \\ g(b, x) & x \end{array}$$

$$f(f(a, \textcolor{red}{f(a, b)}), g(f(a, b), g(b, a)))$$

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Remarque : There is only one "leftmost-innermost" or "parallel-innermost" strategy but many "innermost" strategies.

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Leftmost-outermost strategy

The **leftmost-outermost** strategy rewrites the leftmost redex of all the set of outermost redexes.

$$f(\textcolor{red}{f(a, f(a, b))}, g(f(a, b), g(b, a)))$$

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Parallel-outermost strategy

The **parallel-outermost** strategy rewrites simultaneously the set of ALL the outermost redexes.

$$f(\textcolor{red}{f(a, f(a, b))}, g(\textcolor{red}{f(a, b)}, g(b, a)))$$

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Complete strategy

The **complete** strategy rewrites simultaneously all the redexes.

$$f(\underline{f(a, f(a, b))}, \underline{g(f(a, b), g(b, a))})$$

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Complete strategy (definition)

Let R be an orthogonal system. Define $IF(t) = u$ iff $t \rightarrow_c u$, where

$$\frac{S \text{ in } R\text{-normal form}}{S \rightarrow_c S} \quad (\text{reflexivity}) \quad \frac{l \rightarrow_r R \text{ and } r \rightarrow_c (r)}{(l) \rightarrow_c (r)} \quad (\text{head})$$

$$\frac{S_1 \rightarrow_c t_1 \dots S_n \rightarrow_c t_n}{f(S_1, \dots, S_n) \rightarrow_c f(t_1, \dots, t_n)} \quad (\text{context})$$

and $x \rightarrow_c y$ iff $dom(x) = dom(y)$ and $x \rightarrow_c x \rightarrow_c dom(x)$.

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Are these strategies normalizing?

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The leftmost-innermost strategy is not normalizing

Example : Let R be the following system :

$$\begin{array}{rcl}
 & f(x, b) & d \\
 R & a & b \\
 & c & c \\
 \\
 & f(\underline{c}, a) & f(\underline{c}, a) \quad \dots
 \end{array}$$

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The parallel-innermost strategy is not normalizing

Example : Let R be the following system :

$$\begin{array}{rcl}
 & f(x, b) & d \\
 R & a & b \\
 & c & c \\
 \\
 & f(\underline{c}, \underline{a}) & f(\underline{c}, b) \quad \dots
 \end{array}$$

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In general...

Theorem : Let R an orthogonal system. Then the innermost strategy is normalizing for R iff R is strongly normalizing.

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The leftmost-outermost strategy is not normalizing

Example :

$$\begin{array}{ccc} & f(x, b) & d \\ R & a & b \\ & c & c \\ \\ f(\underline{c}, a) & f(\underline{c}, a) & \dots \end{array}$$

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Parallel Outermost

The **parallel-outermost** strategy is normalizing for orthogonal systems.

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Making the leftmost-outermost strategy normalizing

Définition : A term is **left normal** iff all the function symbols appear before the variables. A system R is **left normal** iff for every rule $l \rightarrow r$ the term l is left normal.

Example :

The term $f(x, b)$ is not left normal, the term $f(b, x)$ is left normal.

Theorem : The **leftmost-outermost** strategy is normalizing for all the orthogonal systems which are **left normal**.

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Complete Strategy

The **complete** strategy is normalizing for orthogonal systems.

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Strategies with history

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Descendants

	$f(g(a))$	$h(g(a), g(a))$
$f(x)$		
$g(a)$		$h(a, g(a))$
	$f(a)$	$h(a, a)$

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Residual theory

J-J. Lévy

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Accidents

$R: I(x) \quad x$

Where does the reduction R take place in $I(I(x)) \quad I(x)$?

$R: a \quad a$

Where does the reduction R take place in $f(a, a) \quad f(a, a)$?

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Solution

We need to specify the rule which is used, and the position in which reduction takes place.

$$f(g(a)) \quad {}_2 f(a)$$

$$f(g(a)) \quad {}_\epsilon h(g(a), g(a)) \quad {}_1 h(a, g(a))$$

$$I(I(x)) \quad {}_\epsilon I(x) \quad I(I(x)) \quad {}_1 I(x)$$

$$f(a, a) \quad {}_1 f(a, a) \quad f(a, a) \quad {}_2 f(a, a)$$

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Formally,

- If p and p are disjoint : $p/ = \{p\}$ so that s appears inside u .
- If p and p are the same : $p/ =$ so that the subterm s is erased in u .
- If $p > p$ (s is a strict subterm of s) : $p/ = \{p\}$, s is an argument of s , so that s appears in u with a different argument.
- If $p > p$ (s is a strict subterm of s) : $p/ =$ the set of all the positions of s in u since s appears $n > 0$ times in u .
- All the other redexes in u are **created**.

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Descendants

Let s and s be two subterms of u , i.e. $u = t[s]_p$ and $u = v[s]_p$. Consider the reduction : $u \rightarrow_p u$ (i.e. s is a redex).

The operation $p/$ specifies the descendants in u of the subterm s at position p of u after the reduction of the redex s at position p of u . Thus, the descendants are given by a set of positions of u .

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Example

$$R \quad \begin{array}{ll} a & {}_1 b \\ b & {}_2 c \\ c & {}_3 d \\ f(x, y) & {}_4 g(y, y) \\ g(x, h(y)) & {}_5 h(y) \end{array}$$

$$t : g(f(c, h(a)), h(b)) \quad {}_4 g(g(h(a), h(a)), h(b)) : t$$

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- The redex a in t is duplicated twice in t .
- The redex b in t appears once in t .
- The redex c in t is erased in t .
- The redex $f(c, h(a))$ in t has no descendant in t .
- The redex $g(f(c, h(a)), h(b))$ in t has a descendant $g(g(h(a), h(a)), h(b))$ in t .
- The redex $g(h(a), h(a))$ in t is **created**.

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Residual of a set of redexes

Let S a set of redexes of t . The set of residuals of S w.r.t. the reduction $\rightarrow_s : t \rightarrow_s t$ is given by

$$S/ = \bigcup_{s \in S} s/$$

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Descendants and residuals

Remarque : Let \mathcal{R} an orthogonal system. Descendants of a redex are always redexes.

Définition : A redex which is descendant of a redex is a **residual**, otherwise it is a **created redex**.

Remarque : Let $\rightarrow : t \rightarrow u$ and $\rightarrow' : t \rightarrow' u$. Then $p/$ and $p'/$ are not necessarily equal. Exemple : $\rightarrow : I(I(x)) \rightarrow I(x)$ and $\rightarrow' : I(I(x)) \rightarrow_1 I(x)$ but $\rightarrow \neq \rightarrow'$ and $\rightarrow' \neq \rightarrow$.

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Residual by a reduction sequence

Définition : Let s be a subterm of t at position p . The **set of residuals** of p w.r.t. the **reduction sequence** $\rightarrow : t \rightarrow t$, written $p/$, is given by :

$$\begin{aligned} p/ &= \{s\} \\ p/ &= s/ , \text{ if } p/ = 1 \\ p/_{0 \ 0} &= (p/_{0})/_{0} \end{aligned}$$

where $/$ is extended to sets as follows :

$$S/ = \bigcup_{s \in S} s/$$

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Developments

A development of a term t is a reduction sequence where no created redex can be reduced, that is, only residual of the set of redex of t can be reduced. Formally,

Définition :

Let S be the set of all the redexes of t . A development of a term t is a reduction sequence :

$$: t \xrightarrow{s_0} t_0 \xrightarrow{s_1} t_1 \xrightarrow{s_2} t_2 \dots$$

such that for every S_i we have $S_i \subseteq S / s_0 \dots s_{i-1}$.

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Complete Developments

Définition : Let S be the set of all the redexes of t . A development $: t \rightarrow t$ is **complete** iff $S / \rightarrow = \emptyset$.

Theorem : Let R be an orthogonal system. Then, the **complete development** strategy is normalizing.

Remarque : The complete strategy and the complete development strategy are the same.

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